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AN EXAMINATION *of various Solutions of* KEPLER'S *Problem, and a ſhort practical Solution of that Problem pointed out. By the Rev.* J. BRINKLEY, A. M. M. R. I. A. *Andrews Profeſſor of Aſtronomy in the Univerſity of* DUBLIN.—*Read Nov. 1ſt,* 1802.

FROM the time when Kepler diſcovered by the obſervations of Tycho Brahe, that the motions of the planets were performed in elliptical orbits, aſtronomers and mathematicians have been continually propoſing different methods of obtaining the true from the mean place of a planet. A direct method of ſolution was long unknown, and even after the improved ſtate of the analytical art furniſhed a direct ſolution, or more ſtrictly ſpeaking, a direct approximation, it was found that indirect ſolutions are the readieſt for practice. Many indirect ſolutions have been given, and I cannot but think that an examination of the principal of them, and a compariſon of their different degrees of accuracy, will be conſidered both as curious and uſeful. It is certainly an object for curioſity to compare together ſolutions, ſome of which brought to their reſpective authors, in their day, no ſmall credit, and ſome of which have ſince been very inaccurately appreciated by different writers; and it is alſo both curious and uſeful to contemplate the preſent ſtate of the analytical art which now ſo readily furniſhes methods of making a compariſon. I know not that this compariſon has before been made, and I have been the more induced to do it, as thereby ſtriking examples are afforded of the advantage of the method of deducing ſeries which I gave in a memoir read at the Academy, Nov. 1798.* The utility of this en-enquiry will be more readily allowed at the preſent time, from the remarkable circumſtance, the diſcovery of two new planets. Heretofore an enquiry of this kind might have appeared of little uſe, as tables for the equation of the centre were already conſtructed for all the planets. Now we actually have two new bodies, one moving in an orbit more excentric than that of Mercury, for which the application of Kepler's problem will be continually neceſſary, till the elements of their orbits are ſettled with preciſion, and tables conſtructed. It alſo may reaſonably be expected that the induſtry of aſtronomers will, ere long, diſcover other bodies of the ſame kind.

The indirect method, which I have here recommended, and which for orbits not very excentric appears to me as ready in practice as can be deſired, is deduced from a combination of the methods of Kepler, Newton, and the ſecond Caſſini; it is alſo applicable to the excentric orbits of comets, and will, in all caſes, rapidly approximate.

The mode of examination of the principal ſolutions, and the reſults from that examination are briefly as follows.

The two ſeries for the true and the excentric anomaly aſcending by the powers of the excentricity, and by the ſeries of multiples of the mean anomaly are firſt given to ſerve as it were for a ſcale to meaſure the true and excentric anomalies deduced from the indirect methods examined.

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* This Memoir was read at the Academy, Nov. 1798, and printed in the ſeventh volume, publiſhed in 1800. In the latter year, a work of conſiderable magnitude in quarto, by M. Arbogaſt appeared, entitled "Calcul des Derivations". The purport of this work is preciſely the ſame as that of my memoir. The method of M. Arbogaſt is, however, very different from mine; but, if I am not miſtaken, my method is ſuſceptible of much more general application, and of greater facility in practice. The limited nature of a memoir prevented me from entering into much detail. I have therefore engaged in drawing up a full and regular account of the method, and of its application to all the purpoſes to which M. Arbogaſt has applied his, ſome of which had not ſuggeſted themſelves, till I ſaw his book, and to other important ones in which his method does not appear to be readily applicable.

Kepler's ſtatement of his problem, and his own ſolution are then noticed.

Boulliald's firſt hypotheſis, and Seth Ward's ſimplification of that hypotheſis are next in order conſidered. Seth Ward's ſimplification, which is better known by the name of Seth Ward's hypotheſis, or the ſimple elliptic hypotheſis, is particularly examined, and the anomaly as deduced thereby, is compared generally with the true anomaly. The ſeries expreſſing his anomaly is remarkable by its ſimplicity, and may be of uſe in other enquiries.

The examination of Boulliald's correction of Ward's hypotheſis follows next. It is ſhewn that this correction, contrary to what has been generally ſuppoſed, at leaſt in more modern times, was a real correction, and afforded a very ſhort and exact method of deriving the true from the mean anomaly. The error is of the ſame order as the third power of the excentricity, and the error of Ward's hypotheſis is of the ſame order as the ſecond power of the excentricity. Obſervations made at that time, could not have detected any error in the places of Venus, the Sun, Jupiter and Saturn; and even in Mars the error, when at its maximum, could ſcarcely have been noticed, except when Mars was then in oppoſition.

In the year 1664, Mercator propoſed a new hypotheſis,* which he, having compared it with forty-two obſerved places of Mars, aſſerted to be more ready in practice, and more accurate than any before given. His hypotheſis is here ſhewn to be ſomewhat leſs accurate than Boulliald's correction of Ward's, although as ready in practice.

Accurately ſpeaking, the above mentioned ſolutions of Ward, Boulliald and Mercator ought not to be called ſolutions of Kepler's Problem. Kepler, reſting upon obſervation, aſſerted that the orbit of a planet was an ellipſe, the ſun being in one of the foci, and alſo that the law of the elliptical motion was ſuch that equal areas were deſcribed in equal times,

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* Aſtr. Edit. 1671, p. 163, 164.

times about the ſun. The problem, therefore, was to aſſign at any time the place of a planet moving according to ſuch a law. Ward, Boulliald and Mercator, however, only adopted the orbit, but not the law of the motion in the orbit. They imagined ſuch a law as would *readily* enable them to deduce the place of the planet, and then had recourſe to obſervation, to eſtabliſh the truth of their reſpective hypotheſes. No one attempted to compare his own hypotheſis with Kepler's diſcovery of the equable deſcription of areas. Each conſidered his own hypotheſis as reſting upon as ſolid foundations as Kepler's. Till the phyſical diſcoveries of the illuſtrious Newton, and the more improved ſtate of aſtronomical inſtruments, it might perhaps have been impoſſible to have decided between the reſpective hypotheſes of Kepler, Boulliald, and Mercator. In reſpect to Ward's Hypotheſis, there could have been no doubt of its imperfection. The ſuperior ſagacity of Kepler, in eliciting from the obſervations of Tycho Brahe the true law, and not reſting upon ſuch a concluſion as Ward has done, can be never ſufficiently admired. However the extreme ſimplicity of the application of Ward's hypotheſis to practice, will always occaſion it to be noticed.

After Mercator's, the two practical ſolutions given by Sir Iſaac Newton* are examined. From the former of theſe two ſolutions a practical one may be derived, which appears to be far preferable to any other that has been given. The ſecond ſolution, although conſidered by its great author, as better adapted for practice, is not ſufficiently exact for the preſent ſtate of aſtronomy, and by extending its accuracy, nothing would be gained in point of brevity.

The next ſolution deſerving notice, was given by the ſecond Caſſini. He pointed out, in 1719,† a very near approximation for the excentric anomaly, and then a correction of that excentric anomaly. The approximation was adopted by De la Caille,‡ and corrected by him nearly in the ſame manner, as Kepler corrects the aſſumed excentric anomaly. Caſſini's corrections of his approximation are very ingenious, but not ſufficient in excentric orbits for the nice purpoſes of modern aſtronomy.

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* Schol. 6 Sect. Lib. 1. Princ. Math.

† Mem. R. Acad. 1719. Caſſini's Aſtr.

‡ De la Caille's Aſtronomy, Art. 144. Vinces Aſt. 225.

This firſt approximation of Caſſini has been adopted by many authors, practice having ſhewn its value; for I know of no one that has attempted to ſhew its exact and general value. It is ſhewn here, and I think for the firſt time, how cloſe an approximation it is, the error depending on the third power of the excentricity. In the orbit of Mars the error is not greater than 20″, and in the orbit of Mercury not greater than 5′. The approximation conſiſts in adding half Seth Ward's anomaly to half the mean anomaly, the ſum will be very nearly the excentric anomaly. The angle Caſſini computes is readily ſhewn to be equal to Seth Ward's anomaly.

It is from this approximation that the method recommended, is partly derived. That method is as follows.

Caſſini's firſt approximation, which is equivalent to the ſum of half Seth Ward's anomaly, and half the mean anomaly, is taken for the excentric anomaly. With this excentric anomaly the mean anomaly is computed by Kepler's method. The difference between this computed mean and the true mean anomaly is multiplied by a number taken out of a ſmall table. This product properly applied to the difference, gives the correction of the approximated excentric anomaly.

The error of the excentric anomaly ſo obtained, is of the ſame order as the ſeventh power of the excentricity, and leſs than a ſecond in all the planets.

The formula from which the table is computed, is derived from Sir Iſaac Newton's firſt method.

If with the corrected approximate excentric anomaly, the operation be repeated, the error of the next approximation will be of the ſame order as the 15th power of the excentricity, and by repeating the proceſſes, the errors will be of the ſame order as the 31ſt, 63d, 127th powers of the excentricity.

Machin has alſo given a ſolution of Kepler's problem,* remarkable for its ingenuity. His motive for attempting the ſolution was, as he tells us, to give one which might be general. None of the methods, according to him, being applicable to excentric orbits, and all of them requiring ſome rule or hypotheſis to begin the computation. He himſelf was aware that his method was too intricate for common practice in regard to the planetary orbits, and had he conſidered Newton's firſt method, he would not have aſſerted that no rule was ſubſiſting, but what was abſolutely uſeleſs in the elliptical orbits of comets. With reſpect to a rule for beginning the computation it may be obſerved that if the mean anomaly be aſſumed for the firſt excentric anomaly, the error of the firſt corrected anomaly will be of the ſame order as the third power of the excentriciy; the ſecond of the ſeventh power, &c. and therefore in the moſt excentric orbits, ſuch an aſſumption would be ſufficient for beginning the computation. Machin's method however, is peculiarly applicable to very excentric orbits. It conſiſts of two parts. By the firſt is obtained an approximation for beginning the computation. The rule, although intricate in practice, is as accurate for very excentric orbits, as for orbits of ſmall excentricity, and in this the merit of the method conſiſts. By the ſecond part, the approximated excentric anomaly is corrected. His method of correction is, as may be readily ſhewn, deduced from a combination of Kepler's and Newton's firſt method; but the author has not acknowledged this circumſtance.

Thomas Simpſon, who ranks ſo high among the Britiſh mathematicians, exerted himſelf on this problem with his uſual ingenuity.† His two firſt methods may be conſidered as illuſtrations of Newton's firſt method. In his third method he obtains the true anomaly, by reverſing the ſeries for expreſſing the mean in terms of the true; and from this ſolution he deduces a ſhort approximation for the true anomaly, but limited, as he obſerves, in point of exactneſs. This practical rule is juſtly commended by Lalande,* as affording a ſhort and accurate method of computation, in orbits not very excentric. It is conſiderably more accurate than either the approximations of Boulliald or Mercator, and nearly as ready in practice as either of their rules. Simpſon compares his rule with the methods of Boulliald and Ward: but by a reference to the paſſage, it will appear that Simpſon miſtook the nature of Boulliald's correction, and therefore he erroneouſly makes Ward's and Boulliald's methods nearly equal in point of accuracy.

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* Phil. Tranſ. Vol. 40. Abridg. vol. 8.

† Simpſon's Eſſays, p. 41. Miſcell. Tracts, p. 46.

In his miſcellaneous eſſays, he computes the correction to be applied to the mean anomaly, to obtain the angle at the higher focus. His motive for attempting this method of ſolution ſeems to have been to ſhew how the accuracy of Newton's ſecond method of finding the anomaly might be farther extended, and alſo to correct ſome miſtaken notions that had been entertained reſpecting Newton's ſolution.

Dr. Matthew Stewart has given alſo a ſolution of Kepler's problem,† the geometrical elegance of the demonſtration of which is to be much admired, as well as the great accuracy of the method, in reſpect to the planetary orbits. His method conſiſts in obtaining an approximation for the excentric anomaly and then correcting that firſt approximation.

The firſt approximation is preciſely the ſame as the above mentioned one of Caſſini. But Stewart's correction of his approximation is more accurate than Caſſini's correction of his approximation. For, as is hereafter ſhewn, the error of the excentric anomaly deduced by Stewarts's method depends, only on the ſeventh and higher powers of the excentricity. This method does not appear to furniſh readily a continued approximation the contrary of which is ſtated in the Life of Stewart, publiſhed in the firſt vol. of the Edinburgh Tranſactions. It may alſo be obſerved, that the firſt approximation, corrected by Newton's method, depends only in the ſeventh and higher powers of the excentricity and the proceſs is ſomewhat more ſimple than by Stewart's method.

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* Lalande's Edit. Halley's Tables, vol. 2. p. 23.

† Edinburgh Tranſactions, 8vo. vol. 3, pag. 105, 120. Stewart's Tracts,

The laſt indirect method which I have noticed, and I know of none of any conſequence later, is De la Caille's.* Lalande recommends this in practice, and has himſelf given a demonſtration of it: But it does not appear to be ſo convenient in practice, as De la Caille's improvement of Caſſini's ſolution, except when the true anomaly is very nearly known; for, as is hereafter ſhewn, the reſpective corrections proceed according to the excentricity, and conſequently in excentric orbits, muſt be repeated many times. Indeed, in examining the accuracy of a table of equations of the centre, this method is very convenient.

I am aware that there are other ſolutions of this problem, not examined here, ſome of which did not require notice, and others, as Lorgna's and Trembley's, cited by Montucla,† I have not ſeen, but ſuppoſe if they had furniſhed any conſiderable improvement they would have been detailed in the laſt edition of Lalande's Aſtronomy.

Of the Series for expreſſing the excentric and true anomaly in terms of the mean anomaly.

Let $m =$ the mean anomaly to radius unity, $e =$ the excentricity of the orbit, the ſemiaxis major being alſo unity, and let c and a repreſent the excentric and true anomalies reſpectively, then

$$c = m - es,\ m + \frac{e^2}{1.2}\ s,\ 2m - \frac{e^3}{1.2.4} \cdot \overline{3s,\ 3m - s,\ m} + \frac{e^4}{1.2.3}\ \overline{2s,\ 4m - s,\ 2m} + \&c.$$

$a = m$

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* Mem. Acad. 1750. Mem. Acad. 1755. Lelande's Ed. Hall Tab.

† Hiſt. Mathem. vol. 2. p. 345.

$$a=m-2es,\ m+\tfrac{5}{4}e^{2}\left.\begin{matrix} s,\ 2m+\tfrac{1}{4}s,\ m \\ -\tfrac{13}{12}s,\ 3m \end{matrix}\right\} e^{3}\left.\begin{matrix} -\tfrac{11}{24}s,\ 2m \\ +\tfrac{103}{96}s,\ 4m \end{matrix}\right\} e^{4}\ \&c$$

Theſe ſeries are here given to ſerve for ſcales of compariſon, as it were, to the different ſeries hereafter inveſtigated from the reſpective hypotheſes examined. It is ſufficient for my preſent purpoſe, merely to ſtate the ſeries without entering particularly into their inveſtigation. They are to be met with in various authors.* I ſhall only obſerve that the ſeries for c, is derived from the equation $c=m\text{-}es,c$. This equation is readily ſolved, and alſo any function of c found by a remarkable and elegant theorem invented by Lagrange,† and afterwards demonſtrated by Laplace. The latter theorem for the true anomaly has been inveſtigated by ſeveral authors by reverſing the ſeries for expreſſing the mean anomaly in terms of the true. It has been deduced by others from firſt finding the excentric from the mean, and then the true from the excentric anomaly. The latter mode is adopted by Laplace, in his incomparable work "Mecanique celeſte." This great mathematician has there given an inveſtigation ſtrikingly elegant. He firſt has obtained, by an ingenious transformation, the law of the ſeries expreſſing the true in terms of the excentric anomaly and excentricity. By combining this concluſion with the ſeries for the excentric anomaly, and the ſines of its multiples, the ſeries for the true anomaly may be continued at pleaſure.

It is hereafter pointed out, how the ſame ſeries for the true anomaly, in terms of the excentric, may be obtained without the introduction of impoſſible quantities. The law of the ſeries, indeed, is not demonſtrated, but only collected by induction; yet it may be a queſtion,

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* Lagrange Berl. Act. 1769. Couſin's Aſt. Phyſ. 43, 44. Laplace Mecanique Céleſte, Liv. 2. c. 3, 22. Lalande Aſt. vol. 3. Tran. R. I. A. vol. 7. 347, 350.

† Berl. Acad. 1768.

tion, in many inſtances, how far the demonſtration of a law by the introduction of impoſſible quantities, exceeds in evidence, a concluſion obtained by induction.

On Kepler's own Solution of his Problem.

Kepler having ſatisfied himſelf, that the orbit of Mars was an ellipſe, and that equal areas were deſcribed in equal times about the Sun, in one of the foci, reduces the problem of finding the coæquate or true anomaly from the mean, to this,[*] "Aream ſemicirculi ex quocunque "puncto diametri in datâ ratione ſecare," and obſerves "mihi ſufficit "credere ſolvi a priori non poſſe propter arcus et ſinus ετερογενιαν. "Erranti mihi quicunque viam monſtrabit, is erit mihi magnus Apol-"lonius."

Accordingly, he himſelf has recourſe to a tentative method of ſolution;[†] he aſſumes the excentric anomaly, and then computes the mean anomaly; the error of the mean anomaly ſo computed, he applies to the firſt aſſumed excentric anomaly, and with the excentric anomaly ſo corrected, he repeats the operation as often as neceſſary. This mode of computing the excentric anomaly is derived from the equation,

$$m = c + c\ s, c$$

which equation follows from the equable deſcription of areas.

From this equation it is evident that the error of the aſſumed value of c differs from the error of the computed value of m, only by a quantity to which c has always a greater ratio than $1 : e$. Therefore, regarding the value of e in all the planets, the excentric anomaly by repeating the operations, rapidly converges to its true value. The excentric anomaly being found, the true anomaly is had by a well known theorem.

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[*] Kepler de Motu Stellæ Martis, p. 300.

[†] Kep. Epitome, Aſt. p. 695.

This ſolution of Kepler's is perhaps, in practical value, little inferior to any that has been ſince given; it obviouſly requires only two improvements, a near approximation to begin the computation, and alſo at once from the error of the computed mean anomaly, to derive the correction of the firſt aſſumed excentric anomaly. The ſecond Caſſini* has given a rule for the former, and applying this rule of Caſſini's to a method given by Sir Iſaac Newton,† a correction is at once obtained, which will give the excentric anomaly, true to leſs than a ſecond in all the planets, as is hereafter ſhewn.

On Boulliald's firſt Hypotheſis and its ſimplification by Seth Ward, commonly known by the name of the ſimple elliptic Hypotheſis.

Kepler's diſcoveries reſted ſimply on obſervations, and on obſervations, which, from the neceſſary imperfection of inſtruments, were liable to errors within certain limits. Any other hypotheſes which would agree with obſervations within theſe limits, were conſidered as equally entitled to notice, as the laws of Kepler. Accordingly, Iſmael Boulliald, one of the greateſt mathematicians of his time, adopted only the elliptic orbit, and not the equable deſcription of areas. Deſirous of deriving the inequable motion in the orbit, from an equable motion, he ſuppoſed " the ellipſe, in which the planet moved to be a ſection of a " certain cone, the axis of which paſſed through the higher focus, and " in which ellipſe, the motion was ſo regulated by ſome phyſical cauſe,

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* Mem. Acad. 1719, † Math. Prin. Nat. Phil. Lib. 1. Sect. 6. Schol.

" that the line joining the planet and vertex of the cone moved with " an equable motion."

This complex hypotheſis Seth Ward ſhewed* to be preciſely the ſame with regard to the motion in the ellipſe, as an equable motion about the higher focus. At the ſame time he gave two methods of computing the true from the mean anomaly on this hypotheſis. One of them was furniſhed by Neil, who is celebrated as the firſt that ever exhibited a right line equal to a curve. This rule of Neil's is the elegant and ſimple one now ſo well known, viz. " As the Aphelion diſtance : Perhelion diſtance : : tang. of half the mean anomaly : tang. of half the true anomaly." Ward afterwards aſſumes, in his Aſtronomia Geometrica, this as the law of a planet's motion, and ſtates himſelf indebted to Boulliald for the hints that led him to the diſcovery. Boulliald is one of thoſe to whom he dedicates his Aſtr. Geo. in that dedication he ſays, " magna certè illius laudis pars in teipſum redundabit, qui " aſtronomiâ philolaicâ me ad hanc rem excitaſti, promoviſti, atque motus " æqualitatem ad axem Coni adeoque (uti in inquiſitione noſtrâ ex " principiis tuis oſtendimus) ad umbilicum alterum ellipſeos referendo adjuviſti."† This appears ſurprizing, when it is conſidered that the ancients conceived an equable motion, about a point within the circular orbit, equally diſtant from the centre as the earth was. The tranſition from this to the upper focus of the ellipſe was obvious. Kepler himſelf remarks it more than once;† but did not attempt to ſhew its connection with the equable deſcription of areas. Ward, which is ſtill more remarkable, conſiders‡ himſelf as having ſolved the problem propoſed by Kepler. How could it eſcape him that the problem he had ſolved was not the problem propoſed by Kepler, unleſs it had been ſhewn that the equable motion about the higher focus reſulted from the equable deſcription of areas?

The method of Ward muſt however, always be conſidered as an elegant and uſeful, although not a near approximation, except when the orbit

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* Inquiſitio in Aſtr. Boull.

† Epit. p. 673, 681.

‡ Præf. ad Aſtr. Geom.

Fig. 1.

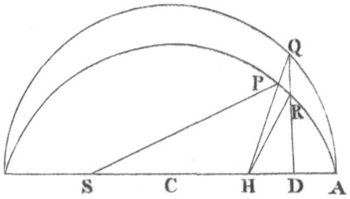


Fig. 2.

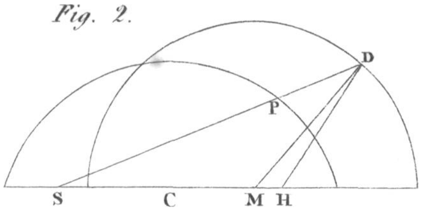


Fig. 3.

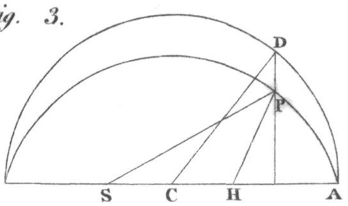


Fig. 4.

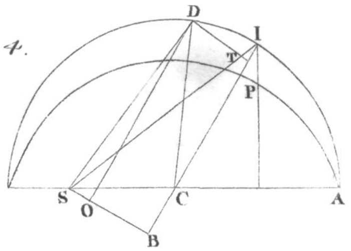


Fig. 5. 1.

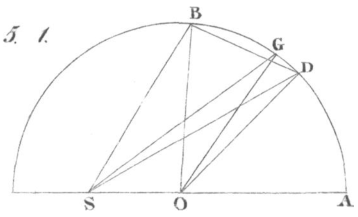
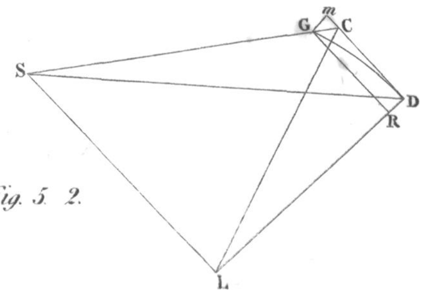


Fig. 5. 2.



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orbit is of ſmall excentricity. It cannot, nevertheleſs, be denied that Ward has obtained more merit than his ſhare in it deſerved.

Let us proceed to enquire its exact value; for this purpoſe the following theorem is neceſſary, which will alſo afterwards be referred to, and may be of uſe in other enquiries.

Theorem.

In a plane triangle, let half the baſe be e, and half the ſum of the ſides be unity, and one of the exterior angles at the baſe $= m$ (rad. unity,) then the oppoſite interior angle will $=$

$$m - \frac{2es,}{1}\, m + \frac{2e^{2}}{2}\, s,\ 2m - \frac{2e^{3}}{3}\, s,\ 3m + \frac{2e^{4}}{4}\, s,\ 4m - \&c. \text{ when } e \text{ is leſs than } 1.$$

Demonſtration. Let a=the interior oppoſite angle,
then by Plane Trig. $1+e : 1-e :: t,\tfrac{1}{2}m : t,\tfrac{1}{2}a$
therefore log. $t,\tfrac{1}{2}a$=log. $t,\tfrac{1}{2}m$—log. $\overline{1+e}$+log. $\overline{1-e}$, or making

e and a vary $\dfrac{\dot{\overline{\tfrac{1}{2}a}}}{t,\tfrac{1}{2}a \times cs^{2},\tfrac{1}{2}a} = \dfrac{2\dot{e}}{1-e^{2}}$ or $\dot{a} \times \overline{1-e}^{2} = -2\dot{e}s,\ a.$

In order to find, from this equation, the value of a correſponding to any values of e and m, in a ſeries aſcending by the powers of e, let the ſucceſſive fluxions of this equation be taken per Saltum,* making $e = o$, and $a = m$.

then $\ddot{a} = 2\dot{e}\dot{a}cs,m$ or $\dfrac{\ddot{a}}{2} = \dfrac{2\dot{e}^{2}}{2} s,2m$, when $\dot{e} = e$

$$\overset{\cdot\,3}{a} - 2\dot{e}^{2}\dot{a} = -2\dot{e}s,\overline{\ddot{a}} \text{ or } \frac{\dddot{a}}{1.2.3} = -\frac{2e^{3}}{3} s,\ 3m$$

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* See Tranſ. R. Iriſh Academy, vol. 7. Mem. p. 321. Prob. 2. & 3.

$$\overset{4}{\dot{a}} - 3\dot{e}^{2}\ddot{a} = -2\dot{e}s,\overline{\ddot{a}}\ \text{or}\ \frac{\overset{4}{\dot{a}}}{1\,.\,.\,4} = \frac{2e^{4}}{4}s,4m$$

$$\overset{5}{\dot{a}} - 12\dot{e}^{2}\overset{3}{\dot{a}} = -2\dot{e}s,\overline{\overset{\cdot 4}{a}}\ \text{or}\ \frac{\overset{5}{\dot{a}}}{1\,.\,.\,5} = -\frac{2e^{5}}{5}s,5m$$

&c. &c. &c.

Hence $a = m + \dot{a} + \frac{\ddot{a}}{2} + \&c. = m - \frac{2e}{1}s,m + \frac{2e^{2}}{2}s,2m - \frac{2e^{3}}{3}s,3m + \&c.$

Q. E. D.

Obſervation. It is well known that

$\sqrt{1+e} : \sqrt{1-e} :: t,\frac{1}{2}c : t,\frac{1}{2}a$, a & c being the true and excentric anomaly of a Planet.

or $t,\frac{1}{2}a = t,\frac{1}{2}c\sqrt{\frac{1+c}{1-e}}$

now if for $\sqrt{\frac{1+e}{1-e}}$ be ſubſtituted $\frac{1+e'}{1-e'}$ and the flux. of the equat. taken.

$\dot{a} \times \overline{1-e}^{2} = -2\dot{e}'s,a$ which equat. is preciſely of the ſame form as that from which the ſeries in the above theorem was derived and therefore in the ſame manner we have

$a = c - \frac{2e'^{1}}{1}s,c + \frac{2e'^{2}}{2}s,2c - \frac{2e'^{3}}{3}s,3c + \frac{2e'^{4}}{4}s,4c - \&c.$

and, becauſe $\sqrt{\frac{1+c}{1-c}} = \frac{1+c'}{1-c'}$, $c' = \frac{c}{1+\sqrt{1-c^{2}}}$. This is the ſame concluſion as Laplace has deduced* by a very ingenious uſe of impoſſible quantities.

From the above theorem it immediately follows that the anomaly computed by Seth Ward's method.

$= m - 2es,m + \frac{2c^{2}}{2}s,2m - \frac{2c^{3}}{3}s,3m + \frac{2c^{4}}{4}s,4m - \&c.$

comparing this with the true anomaly,

$m -$

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* Mecanique celeſte, tome 1. p. 170.

$$m - 2es,m + \tfrac{5}{4}e^{2}\,s,\,2m + \left.\begin{array}{l} \tfrac{1}{4}s,\,m \\ -\tfrac{13}{12}s,\,3m \end{array}\right\} e^{3} \left.\begin{array}{l} -\tfrac{11}{24}s,\,2m \\ +\tfrac{103}{96}s,\,4m \end{array}\right\} e^{4}\ \&c.$$

the correction to be applied to Seth Ward's anomaly appears to be

$$\tfrac{1}{4}e^{2}\,s,\,2m + \left.\begin{array}{l} \tfrac{1}{4}s,\,m \\ -\tfrac{5}{12}s,\,3m \end{array}\right\} e^{3} \left.\begin{array}{l} -\tfrac{11}{24}s,\,2m \\ +\tfrac{55}{96}s,\,4m \end{array}\right\} e^{4}\ \&c.$$

The maximum of this correction in the orbit of Mercury, amounts to 33 nearly. In the orbit of Mars to nearly ′7½, a quantity readily detected by the obſervations of Tycho Brahe; and hence this hypotheſis was juſtly diſregarded by Kepler.

On Boulliald's correction of Seth Ward's Hypotheſis.

Boulliald, diſpleaſed at the treatment he had received from Ward, in return ſhewed that the hypotheſis of equable motion about the higher focus, would not give the places of Mars agreeably to obſervation, and propoſed the following conſtruction, for determining the place of a planet. A conſtruction admitting of an eaſy calculation.

Make the angle A H R = mean anomaly (*m*), H being the higher focus, draw D R Q parallel to the axis minor, and meeting the circle in Q, and draw Q P H interſecting the orbit in B, which will be the place of the planet. Fig. 1.

To compare the anomaly ſo determined with the true anomaly, the value of P S H (*a*) is to be obtained in a ſeries aſcending by the powers of the excentricity.

Let Q H A = *z*, then

tang. z : tang. m : : 1 : $\sqrt{\overline{1-e}^{2}}$ or log. t, z=log. t, $m - \tfrac{1}{2}$ log. $\overline{1-e}^{2}$ from which equation, by taking the ſucceſſive fluxions per Saltum, z will be obtained. When $e=0$ and $z=m$ and $\dot{e}$ is conſtant.

1. $\dot{z}=0$

2. $\ddot{z}=\frac{1}{2}\dot{e}^{2}s,\ 2m \qquad \frac{\ddot{z}}{2}=\frac{e^{2}}{4}s,\ 2m$

3. $\overset{3}{\dot{z}}=0$

4. $\overset{4}{\dot{z}}-2.3\dot{e}^{2}\ddot{z}=\underset{2}{3\dot{e}^{2}}\,\overline{s,2\ddot{z}}$ or $\frac{\overset{4}{\dot{z}}}{1.4}=-\frac{1e^{4}}{32}s,\ 4m$ &c.

Therefore $z=m+\frac{1}{8}e^{2}s,\ 2m-\frac{1}{32}e^{4}s,\ 4m$ &c.

and from thence $s,\ z=s,\ m+\frac{1}{8}e^{2}\,\overline{s,\ 3m+s,\ m}$

$s,\ 2z=s,\ 2m+\frac{1}{4}e^{2}s,\ 2m$

$s,\ 3z=s,\ 3m$

$s,\ 4z=s,\ 4m$

&c. &c.

But by the above theorem

The < P S H= $z-\frac{2es,\ z}{1}+\frac{2e^{2}}{2}s,\ 2z-$&c.

in which by ſubſtituting for z s,z &c.

The < P S H or Boulliald's anomaly

$$=m-2es,\ m+\frac{5}{4}e^{2}s,\ 2m\left.\begin{matrix}-\frac{1}{4}s,\ m\\-\frac{13}{12}s,\ 3m\end{matrix}\right\}e^{3}\left.\begin{matrix}+\frac{1}{8}s,\ 2m\\+\frac{25}{32}s,\ 4m\end{matrix}\right\}e^{4}\text{ \&c.}$$

comparing this with the true anomaly, its correction will be found to be

$$\left.\begin{matrix}\frac{1}{4}s,\ m\\-\frac{1}{6}s,\ 3m\end{matrix}\right\}e^{3}\left.\begin{matrix}-\frac{7}{12}s,\ 2m\\-\frac{7}{24}s,\ 4m\end{matrix}\right\}e^{4}\text{ \&c.}$$

The maximum of this correction in the orbit of Mercury, is nearly 20′; in that of Mars nearly 2′, a quantity ſcarcely diſcoverable by the obſervations of Tycho Brahe. In the orbit of Jupiter the greateſt error is nearly 16″, in the earth 0″. So that this method of computing the anomaly, was, in reſpect to the obſervations by which Boulliald examined his hypotheſis, ſufficiently accurate for all the planets except Mercury. It appears alſo, from what has been done, that the maximum of error, in Ward's hypotheſis : max. of error in Boulliald's : : $3 : 8e$ nearly when e is ſmall. Hence, for all the planets the hypotheſis of Boulliald is more accurate then that of Ward, and by no means deſerves the cenſure that has been paſſed upon it, by ſome authors. Emerſon obſerves,[*] the correction of Seth Ward's hypotheſis given by Boulliald, only mends it in ſome places, and in others makes it worſe. Simpſon, as I obſerved, miſtook this correction. He ſuppoſed[†] Q H A to be the mean anomaly, and R the place of the planet. On this ſuppoſition he found the error much the ſame as the error of Seth Ward's anomaly. Montucla[‡] ſays, that Boulliald's hypotheſis "*ne vaut pas mieux.*"

Indeed, it is remarkable how much Boulliald has been miſrepreſented.

Adam Smith, in his Hiſtory of Aſtronomy[‡] ſays "Thus Boulliald, who cenſured this hypotheſis of Ward, invented another of "the ſame kind, infinitely more whimsical and capricious." Then he proceeds to give, not the ſecond hypotheſis of Boulliald, but the firſt hypotheſis to which Ward had ſhewn his own hypotheſis equivalent.

Of Mercator's Hypotheſis.

Mercator, not ſatisfied with either of the hypotheſes above mentioned, invented what he calls his new hypotheſis, which he conſidered as more accurate than any preceding one. With what juſtice, I ſhall proceed to examine, by comparing his anomaly with the true anomaly. He himſelf compared it with forty-two places of Mars obſerved by Tycho Brahe.

His hypotheſis was as follows,

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[*] Emerſon's Aſtron. p. 208.

[†] Eſſays, p. 48.

[‡] Hiſt. Math. Tom. 2. Edit. 2, page 340.

Let the diſtance between the foci be divided in extreme and mean Fig. 2. ratio, about the dividing point, (nearer the higher focus) let a circle be deſcribed with a radius, equal the ſemiaxis major; from the upper focus draw a line interſecting the circle and making with the axis major an angle = the mean anomaly; and join the point of interſection, and the lower focus by a line, which will cut the ellipſe in the place of the planet, according to Mercator.*

Now if M be the point of diviſion, let MH= $\overline{(3-\sqrt{5}\times e)}=ne$.

Let D S H= a, and M D H=x

then 1. $s,x=nes,m$

and s, H : s, SDH : : SD : SH

s, DSH : s, HMD : : MD : SD

Therefore $s,m\times s,a : s,\overline{m-a}\times s,\overline{m-x} :: 1 : 2c$

or 2. $s,\overline{m-a}\times s,\overline{m-x}=2es,\overline{m\times s,a}$

Let the ſucceſſive fluxions of theſe equations be taken *per Saltum*.

When $a=m$, $x=o$, $e=o$ and $\dot{e}$ is conſtant we have

1. $\dot{x}=n\dot{e}s,m$ $\qquad -\dot{a}=2\dot{e}s,m$

2. $\ddot{x}=o$ $\qquad -\ddot{a}s,m+2\dot{a}\dot{x}cs,m=2sm\times2\dot{a}\dot{e}cs,m$

3. &c. $\qquad -\dddot{a}+\dot{a}^{3}+3\ddot{a}\dot{x}cs,m+3\dot{a}\dot{x}^{2}s,m=$

$$=2.3\dot{c}\ddot{a}s,m\times cs,m-2\dot{c}\dot{a}^{2}\,{}^{2}s,m$$

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* Montucla, Hiſt. Math. Tom. 2, p. 340, has ſtated this hypotheſis very imperfectly and inaccurately: his words are, "Nicoles Mercator y fit dans le ſuite, une correc-"tion. Il partagea la diſtance entre les foyers de l'ellipſe in moyenne et extrême "raiſon, de ſorte que le point de ſection tombât au delà du centre á l'egard du foyer "occupé par la planète centrale, et ce fut *ce point qu'il prit pour centre du movement* "*moyen.*"

Hence reducing theſe equations,

$$a = m + \dot{a} + \frac{\ddot{a}}{1.2} + \&c.$$

$$= m - 2esm + 1{,}23607e^{2} s,\ 2m + 0{,}79837e^{3} s,\ m - 1{,}28407e^{3} s,\ 3m\ \&c.$$

but true anom. $= m - 2esm + 1{,}25e^{2} s,\ 2m + 0{,}25e^{3} s,\ m - 1{,}08333e^{3} s,\ 3m$ &c

Therefore the error of Mercator's anomaly

$$= -01393e^{2} s,\ 2m + 0{,}54837e^{3} s,\ m - 0{,}20137e^{3} s,\ 3m\ \&c.$$

The greateſt error in the orbit of Mercury will be 22′½ nearly, and in the orbit of Mars 2′ 7″. Hence this hypotheſis of Mercator gives the place of a planet ſomewhat leſs accurately than Boulliald's correction of Ward's hypotheſis.

The following Table ſhews nearly the maxima of the errors of the three hypotheſes of Ward, Boulliald and Mercator, for all the planets.

	Mercur.	Venus	Earth	Mars	Ceres	Pallas	Jupit.	Saturn	Georg.
Ward	33′.	2″	15″	7¼′	5¼′	57′	2′	2½′	2′
Boulliald	20′	0	1″	1′. 52″	1′. 8″	39′. 49″	16″	24″	14″
Mercator	22½′	0	1″	2′. 7″	1′. 19″	45′	19″	29″	16″

The mean anomalies, when the reſpective errors of theſe hypotheſes are greateſt, may be found ſufficiently near as follows. The errors deduced will in Mercury and Pallas, differ a few ſeconds from the truth. In Seth Ward's hypotheſis $m = ,7854 + \frac{3}{2\sqrt{2}} e + \frac{1}{3} e^{2}$. In Boulliald's $m = 90°$ and in Mercators, the coſine of $m = -\frac{,0117}{e}$. This laſt expreſſion evidently will not apply to the Earth and Venus. For them, ſtrictly, the ſolutiou of a cubic equation is neceſſary, but the limit of the error is otherwiſe obvious.

On

On Newton's two practical methods of ſolving Kepler's Problem.

Both theſe methods* are given without demonſtration. The former conſiſts in obtaining ſucceſſive approximations to the excentric anomaly. Keil has given a demonſtration† of it, but has not aſſigned the rate of convergence of the Newtonian ſeries, from which ariſes the great value of this method.

It may be explained and demonſtrated from the conſideration of the equation $c=m-e\,s,c$ by which the rate of converging will be pointed out.

Let a near aſſumed value of c be $\overset{\prime}{c}$ and let $c=\overset{\prime}{c}+\underset{\prime}{c}$

let alſo $\overset{\prime\prime}{c}=m-es,\overset{\prime}{c}$

then becauſe $\overset{\prime}{c}+\underset{\prime}{c}=m-es,\overline{\overset{\prime}{c}+\underset{\prime}{c}}$

we have $\overset{\prime}{c}+\underset{\prime}{c}-\overset{\prime\prime}{c}=e\times\overline{s,\overset{\prime}{c}-s,\overline{\overset{\prime}{c}+\underset{\prime}{c}}}$

now $s,\overline{\overset{\prime}{c}+\underset{\prime}{c}}=s,\overset{\prime}{c}\times cs,\underset{\prime}{c}+cs,\overset{\prime}{c}\times s,\underset{\prime}{c}$

but $cs,\underset{\prime}{c}=1-\frac{1}{2}\underset{\prime}{c}^{2}+$&c. and $s,\underset{\prime}{c}=\underset{\prime}{c}-\frac{1}{2.3}\underset{\prime}{c}^{3}+$&c.

Therefore $\overset{\prime}{c}+\underset{\prime}{c}-\overset{\prime\prime}{c}=-e\underset{\prime}{c}\times cs,\overline{\overset{\prime}{c}-\tfrac{1}{2}\underset{\prime}{c}s,\overset{\prime}{c}}$ not regarding $\underset{\prime}{c}^{3}$ &c.

$$\text{or } \underset{\prime}{c}=\frac{\overset{\prime\prime}{c}-\overset{\prime}{c}}{1+e\times\overline{csc-\tfrac{1}{2}\underset{\prime}{c}sc}}=\frac{\overset{\prime\prime}{c}-\overset{\prime}{c}}{1+ecs,\overset{\prime}{c}}\text{ nearly.}$$

Newton's

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[*] Prin. Math. Nat. Phil. Lib. 1. Sect. 6, Schol.

[†] Phil. Tranſ. Keil's Aſt. 25 Sect. Horſley's Newt. vol. 2, page 133.

Newton's firſt approximation is $\underset{\backslash}{c}=\dfrac{\overset{//}{c}-\overset{/}{c}}{1+ecs,\overset{/}{c}}$, or rather

$\underset{\backslash}{c}:\overset{\backslash\backslash}{c}-\overset{\backslash}{c}::\dfrac{1}{e}:\dfrac{1}{e}+cs\overset{\backslash}{c}$, as will appear by a reference to his rule and if inſtead of

$\overset{\backslash}{c}$, $\overset{\backslash}{c}+\underset{\backslash}{c}$ thus computed be uſed, and then $\overset{\backslash\backslash\backslash}{c}=m-cs,\overline{\overset{\backslash}{c}+\underset{\backslash}{c}}$ and $\underset{\backslash\backslash\backslash}{c}=\dfrac{\overset{\backslash\backslash\backslash}{c}-\overset{\backslash}{c}-\underset{\backslash}{c}}{1+ecs,c+\underset{\backslash}{c}}$

be computed $c+\underset{\backslash}{c}+\underset{\backslash\backslash}{c}$ will be a nearer approximation to c, which may be continued at pleaſure.

With reſpect to the rate of convergency of the quantities

$\overset{\backslash}{c}+\underset{\backslash}{c}$, $\overset{\backslash}{c}+\underset{\backslash}{c}+\underset{\backslash\backslash}{c}$, &c. to c.

We have $\underset{\backslash}{c}=\dfrac{\overset{\backslash\backslash}{c}-\overset{\backslash}{c}}{1+ecs\overset{\backslash}{c}}\times 1-\dfrac{\frac{1}{2}ecs\underset{\backslash}{\overset{\backslash}{c}}.}{1+ecs\overset{\backslash}{c}}$ Now if the error of $\overset{\backslash}{c}$ be of the ſame order as the firſt power of the excentricity, the quantity $\overset{\backslash\backslash}{c}-\overset{\backslash}{c}$ may be of the ſame order and ∴ the error in $\underset{\backslash}{c}$ ariſing from neglecting $\dfrac{\overset{\backslash\backslash}{c}-\overset{\backslash}{c}}{1+ecs\overset{\backslash}{c}}\times\dfrac{\frac{1}{2}ecs\underset{\backslash}{\overset{\backslash}{c}}}{1+ecs\overset{\backslash}{c}}$ may be of the order of the third power, becauſe $\underset{\backslash}{c}$ and e are of the firſt power.

Hence aſſuming the excentric anomaly = the mean anomaly, the error after the firſt Newtonian operation cannot be of a higher order than the third power of the excentricity, becauſe the difference between the mean and excentric anomaly = $e\ s,c$. By the ſame reaſoning, if the excentric anomaly corrected by the firſt operation, be uſed, the error in the ſecond corrected excentric anomaly will be of the ſame order as the 3 + 3 + 1 = 7th power of the excentricity. The error of the next will be of the 15th, &c.

Hence ſuppoſing the error of c of the order of the firſt power of excentricity the error of $\overset{\backslash}{c}+\underset{\backslash}{c}$ will be of the order of 3d. power.

of $\overset{\backslash}{c}+\underset{\backslash}{c}+\underset{\backslash\backslash}{c}$ of 7th.

of $\overset{\backslash}{c}+\underset{\backslash}{c}+\underset{\backslash\backslash}{c}+\underset{\backslash\backslash\backslash}{c}$ of 15th.

of $\overset{\backslash}{c}+\underset{\backslash}{c}+\underset{\backslash\backslash}{c}+\underset{\backslash\backslash\backslash}{c}+\underset{\backslash\backslash\backslash\backslash}{c}$ of 31ſt.

Hence this method is general for any elliptic orbit, however excentric, as Keil juſtly obſerves.*

On Newton's ſecond Method.

This method conſiſts in finding the quantity to be applied to the mean anomaly, to find the angle at the higher focus, which being known, the angle at the ſun is had by the common proportion. The two corrections given by Newton, are of the orders of the ſecond and third powers of the excentricity, and the higher powers are neglected; conſequently this method will not be ſufficient for the orbit of Mercury, and to correct it farther, by extending the terms, would require the ſame trouble as computing directly by the ſeries, the true from the mean anomaly; ſo that this method offers nothing to be particularly remarked.

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* Newton's Anomaly is reckoned from Perihelion.

However, as commentators have laboured much in inveſtigating* Newton's correction, it may not be improper to give the following eaſy method of deriving terms equivalent to the terms of the Newtonian ſeries, which may, at the ſame time, ſerve to ſhew that the inveſtigation of the angle at the higher focus muſt be as tedious as the direct method of finding the angle at the ſun.

Let the angle ſought AHP$=w$, DCA$=c$, the mean anomaly$=m$, Fig. 3. AC$=1$ and CH$=e$. Then from the equable deſcription of areas,

$$m=c+es,\ c$$

Alſo by the well known theorem for finding the true from the excentric anomaly.

$\sqrt{1+e} : \sqrt{1-c} :: \text{tang. } \tfrac{1}{2}\text{SCD} : \text{tang. } \tfrac{1}{2}\text{SHP}$

But tang. $\tfrac{1}{2}$SCD$=$tang. $\frac{180-\text{DCA}}{2}=$co-tang. $\tfrac{1}{2}$DCA

and tang. $\tfrac{1}{2}$SHP$=$co-tang. $\tfrac{1}{2}$PHA

Therefore $\sqrt{1+e} : \sqrt{1-e} ::$ co-ta. $\tfrac{1}{2}c$: co-ta. $\tfrac{1}{2}w$ :: ta. $\tfrac{1}{2}w$: ta. $\tfrac{1}{2}c$

or log. tang. $\tfrac{1}{2}w=$log. tang. $\tfrac{1}{2}c+\tfrac{1}{2}$log. $\frac{1+e}{1-e}$

Taking the fluxions of the equations thus obtained, making only e, c and w variable, we have

$$\left.\begin{array}{l}\dot{c}+\dot{e}s,\ c=0\\[4pt] \dfrac{\dot{w}}{s,\ w}=\dfrac{\dot{c}}{s,\ c}-\dfrac{\dot{c}}{1-c^{2}}\end{array}\right\}$$

From which equations the reſpective values of w, $\dot{w}$, $\ddot{w}$, &c. are to be obtained when $e=0$ and $c=w=m$ by taking the ſucceſſive fluxions of theſe equations per Saltum.

$$1.\ \left.\begin{array}{l}\dot{c}+\dot{e}s,\ m=0\\ \dot{w}=\dot{c}-\dot{e}sm\end{array}\right\}\text{therefore } \dot{c}=-\dot{e}sm \text{ and } \dot{w}=0$$

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* Simpſon's Tracts, pag. 49. Horſley's Newton, vol. 2. p. 135. Emerſon's Comment. Le Sœur & Jacquier.

$$2.\quad \left.\begin{array}{l}\ddot{c}+2\ddot{e}ccs,\,m=0\\[6pt] \ddot{w}=\dot{c}-\dot{c}^{2}\dfrac{csm}{s,\,m}\end{array}\right\}\text{therefore }\ddot{c}=\dot{e}^{2}s,\,2m\text{ and }\ddot{w}=\tfrac{1}{2}\dot{e}^{2}s,\,2m$$

$$3.\quad \left.\begin{array}{l}\overset{3}{\dot{c}}+3\dot{e}\times\overline{\ddot{c}cs,\,m-\dot{c}^{2}s,\,m}=0\\[6pt] \overset{3}{\dot{w}}=\overset{3}{\dot{c}}-\dfrac{3\dot{c}\dot{c}\dot{c}sm}{s,\,m}-\dot{c}^{3}\dfrac{+2\dot{e}^{3}cs,{}^{2}m}{s^{2},\,m}\end{array}\right\}\text{therefore }\overset{3}{\dot{w}}=4\dot{e}^{3}s^{3}\,m=\overline{3sm-s,\,3m}\,\dot{e}^{3}$$

Hence $w=m+\frac{1}{4}e^{2}\,s,\,2m+\frac{1}{2}e^{3}\,.\,\overline{s,\,m-\frac{1}{3}s,\,3m}+\&c.$
which terms are equivalent to the two corrections applied by Sir Iſaac Newton.

*On the ſecond Caſſini's method.**

The method of Caſſini, now in order to be examined, is perhaps both the moſt elegant that has yet been invented, and the moſt readily Fig. 4. deduced from the equable deſcription of areas, and is alſo, in the planetary orbits, of very great practical value.

Let P be the place of the planet, and AI the excentric anomaly. SB perpendicular to CI. Let AD = the mean anomaly, and draw DT perpendicular, and DO parallel to CI. Then becauſe DA is the mean anomaly, the area DCA = SAI = SIC + ICA, and therefore SCI = DCI conſequently SB = the arch DI.

Hence three approximations are derived by Caſſini. 1. SD is nearly parallel to CI, therefore if the angle SDC be computed, DCI will be found nearly, and conſequently ACI the excentric anomaly.

2. SO = DI — DT. And therefore when the excentricity is not great, the meaſure of the angle SDO=DI—DT nearly. This latter quantity being taken from SDC, the angle DCI is obtained.

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* Mem. Acad. 1719. Caſſini Elem. Aſtr. vol. 1. p. 143.

3. SD : SO=DI–DT : : rad : s, SDO. Hence, by computing SD, a more accurate value of SDO will be had, and therefore a more accurate value of ODC=DCI.

TO obtain the ſecond approximation, Caſſini gives a table containing the difference of the arches and their ſines for a few of the firſt degrees of the quadrant, a table which is eaſily extended.

The reſpective degrees of accuracy of theſe approximations may be inveſtigated as follows.

1. We have $CD+SC : CD-SC : : t,\ \frac{1}{2}\angle DCA : t,\ \frac{\angle DSC-\angle SDC}{2}$

or $1+e : 1-e : : t,\ \frac{1}{2}m : t,\ \frac{1}{2}w$

which is the ſame proportion as that in the preceding theorem,

whence $w=m-\frac{2}{1}es,\ m+\frac{2}{2}e^{2}s,\ 2m-\frac{2}{3}e^{3}s,\ 3m+\&c.$

Indeed it is evident the angle ∠DSC—∠SDC= Seth Ward's anomaly. For the triangle SCD is ſimilar to the triangle formed by the diſtance between the foci and the axis major, including an angle equal to the mean anomaly according to Seth Ward's hypotheſis.

Hence $SDC=es,\ m-\frac{1}{2}e^{2}s,\ 2m+\frac{1}{3}e^{3}s,\ 3m-\&c.$

But in Caſſini's firſt approximation DCI is taken equal to SDC therefore according to that approximation ACI the excentric anomaly $=m-es,m+\frac{1}{2}e^{2}s,\ 2m-\frac{1}{3}e^{3}s,\ 3m+\&c.$ or the excentric anomaly$=\frac{1}{2}$ the mean anomaly$+\frac{1}{2}$ Seth Ward's anomaly. Taking the difference between this and the true excentric anomaly we have the error of Caſſini's firſt approximation $=\frac{1}{24}e^{3}\cdot\overline{3sm-s,\ 3m}\ (\frac{1}{6}e^{3}s^{3},\ m)+e^{4}\cdot\overline{\frac{1}{12}s,\ 4m}-\overline{\frac{1}{6}s,\ 2m}$ &c. in which the third and higher powers of the excentricity only are concerned.

This quantity, when a maximum in the orbit of Mercury, is 5′ nearly, in the orbit of Mars only 20″ &c.

The facility with which this near approximation may be obtained, renders it highly valuable, when combined with the method of extending at pleaſure the approximation. It alſo deſerves notice, from the elegance of the conſtruction ſo readily deduced from the equable deſcription of areas.

2. If SDO could be accurately computed, DCI, and therefore the excentric anomaly, would be had directly. But SO is computed from taking SB—BO= DI—DT, when DCI is taken equal to SDC, and hence one ſource of error. Another ſource of error ariſes from taking SD=DC, and therefore SO for the meaſure of the angle SDO to the radius DC

The computed value therefore of SDO to radius CD=SO=SB—BO= DI—ſine DI=$\frac{DI^3}{2.3}$—&c. $=\frac{1}{6}e^3 s^3, m-\frac{1}{4}e^4 s^2, m\times s, 2m$ &c. conſequently the computed value of ACI=ACD—DCI=$m-es, m+\frac{1}{2}e^2 s, 2m-e^3 \cdot \overline{\frac{3}{8}s, 3m-\frac{1}{8}s, m}+e^4 \cdot \overline{\frac{1}{6}s, 4m-\frac{1}{3}s, 2m}$—&c. this quantity is leſs than the excentric anomaly by $e^4 \cdot \overline{\frac{1}{48}s, 4m-\frac{1}{24}s, 2m}$ &c. The maximum of which is when $m = 60°$ or 120°, and therefore in the orbit of Mercury is, regarding only the fourth power of the excentricity, about 20″. In the orbit of Mars, and all the planets except Pallas, leſs than a ſecond.

This ſecond approximation is almoſt as readily derived, as the firſt, conſidering how eaſily the arch equal to the difference between an arch and its ſine may be obtained.

The true value of DS being uſed the error of SDV ariſes only from the error of SO. But becauſe SO=$\frac{DI}{2.3}$—&c. and that the error of DI depends only on the third and higher powers of the excentricity DI itſelf depending on the firſt power, it follows that the error of SO will depend only on the fifth and higher powers of the excentricity. In practice however, it is more convenient to uſe only the firſt or ſecond approximation, and then if the excentricity of the orbit requires it, to obtain a farther correction, by the method hereafter pointed out.

On

*On Machin's Method.**

The accuracy and convenience of the methods hitherto examined excepting the firſt method of Newton, depend on the ſmall excentricity of the orbit; and therefore they are only applicable to the planets. They would be uſeleſs for the excentric ellipſes in which the comets move. But the problem is otherwiſe ſolved for the parts of the orbits in which the comets are viſible to us. That a complete ſolution to Kepler's problem might be given, equally applicable to all orbits, Machin propoſed his very ingenious method, which does not at all depend on the excentricity. For the planetary orbs indeed the method as given by the author, is not ſo convenient for practice as other methods. However his method may be rendered conſiderably more ſimple in practice, as is hereafter pointed at. In examining this method, the method itſelf is firſt briefly and ſomewhat more ſimply ſtated, than is done by the author in the Phil. Trans. a limit of the error of the firſt approximation is then ſhewn, and alſo the rate of convergency of the ſecond approximation.

The firſt approximation may be explained as follows.

Let m = the mean anomaly (rad. 1), c = the excentric anomaly reckoned from perihelion, for a reaſon to be hereafter aſſigned, the ſemiaxis major $= 1$ and the excentricity $= e$.

Then $m = c - es,c$, let $c = na$, a being an arch, the ſine of which is s and n a number to be determined hereafter.

Then $m = na - es, na$.

Therefore by the ſeries for the expreſſing the arch in terms of the ſine, and by the ſeries for expreſſing the ſine of the multiple arch.

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* Phil. Trans. vol. 40. Abridg. vol. 8.

$$m = \begin{cases} n\times : s + \frac{s^3}{2.3} + \frac{3\cdot 3}{2\cdot3\cdot4\cdot5}s^5 + \frac{1\cdot3\cdot3\cdot5\cdot5}{2\cdot3\cdot\cdot7}s^7 + \&c. \\ en\times : -s + \frac{\overline{n^2-1}}{2\cdot3}s^3 - \frac{\overline{n^2-1}\cdot\overline{n^2-3}^2}{2\cdot3\cdot4\cdot5}s^5 + \frac{\overline{n^2-1}\ldots\overline{n^2-5}^2}{2\cdot3\cdots\cdot7}s^7\ \&c. \end{cases}$$

If $e\times\overline{n^2-1}\,.\,\overline{n-3}^2=9$, the co eff. of $s^5=0$ and $n=\sqrt{5+\sqrt{\frac{9}{e}+16}}$ now as e cannot be greater than unity, this value of n can never be leſs than $\sqrt{10}$ and therefore with it s (=ſine $\frac{c}{n}$) will be always ſo ſmall that the above ſeries will ſwiftly converge.

The equation becomes taking this value of n.

$$\frac{m}{n} = \overline{1-e}\ s + .\frac{\overline{\overline{n^2-1}.\ e+1}}{2\cdot3}s^3 + \frac{n^2}{560}s^7 + \&c.$$

neglecting the terms after s^3 the value of s is had nearly by the reſolution of a cubic equation, and s being found na or c is thence computed for the firſt approximation.

Had the motion been reckoned from aphelion, the value of n would have been $\sqrt{5+\sqrt{\frac{9}{e}-16}}$, and therefore would not have been generally poſſible.

To correct this approximated excentric anomaly *ad libitum*, Machin computes by it the mean anomaly correſponding, by Kepler's rule, and the error of the mean anomaly ſo computed, he divides by the planets diſtance from the ſun, the quotient is the correction to be applied to the approximated excentric anomaly, to obtain a ſecond approximation, &c.

The author ſhews by ſeveral examples that in orbits of great and ſmall excentricity the firſt approximation is always ſufficient to apply the ſecond rule with advantage.

This will be ſeen generally by the following conſiderations by which a limit of the error of the firſt approximation is obtained for any value of c.

The two ſeries above given each converge ſloweſt for a given value of c when n is leaſt, becauſe then $s\ (=s,\ \frac{c}{n})$ is greateſt. The terms omitted therefore bear a greater proportion to the terms retained, and conſequently the error of the value of s determined from the cubic equation is then greateſt for a given value of c. We derive then the general limit of the error of any value of c, by taking $e=1$ and therefore $n=\sqrt{10}$. The equation becomes $m=\sqrt{10}\times:\frac{10}{6}s^{3}+\frac{s^{7}}{56}+$ &c. Now it will eaſily appear by conſidering the formation of the co-efficients of the two ſeries in the caſe of $n=\sqrt{10}$ that the terms $\frac{s^{7}}{56}+$&c. are all poſitive and their co-efficients converging. Hence the ſum of the terms $\frac{s^{7}}{56}+$&c. omitted is leſs than $\frac{1}{56}\times\frac{s^{7}}{1-s^{2}}$. But if the cubic equation

$m=\sqrt{10}\times\frac{10}{6}s^{3}$ be varied by adding $\frac{\sqrt{10}}{56}\times\frac{s^{7}}{1-s2}$ to the right hand ſide, the variation of s will be nearly $=-\frac{1}{280}\times\frac{s^{5}}{1-s^{2}}$ and therefore of

$na=-\frac{1}{280}\times\frac{s^{5}}{1-s^{2}}\times\frac{n}{cs,\,a}=-\frac{\sqrt{10}}{280}\times s^{2}\times{}^{3}t,\,a$. This quantity then will be always greater than the error of the firſt approximated excentric anomaly computed by the above method. When the mean anomaly is 180°, s is greateſt, and then this quantity = 1° 40′ nearly. The actual error by computation appears to be 1°. 13′. When the excentric anomaly is leſs than 23°, the above limit of the error will be leſs than a ſecond; ſo that in every elliptic orbit, when the excentric anomaly is leſs than 23°, the above method will ſuffice, without further correction, to find the excentric anomaly to leſs than a ſecond.*

The demonſtration of the rule for a farther correction *ad libitum*, the author has not given, nor ſhewn the convergency of the ſucceſſive corrections.

To demonſtrate the rule. Let $\grave{m}$ be the mean anomaly computed from the approximated excentric anomaly $\acute{c}$ let $\acute{m}+\underset{\prime}{m}$ and $\grave{c}+\underset{\backprime}{c}$ be the accurate mean and excentric anomalies.

Then $\grave{m}+\underset{\backprime}{m}=\acute{c}+\underset{\backprime}{c}-es,\overline{c+\underset{\backprime}{c}}$ and $\grave{m}=\grave{c}-es,\acute{c}$ therefore

$\underset{\backprime}{m}=\underset{\backprime}{c}-\underset{\backprime}{c}ecs,\grave{c}+\frac{1}{2}\underset{\backprime}{c}^{2}es,\grave{c}$ nearly, neglecting the powers of $\underset{\backprime}{c}$ above the ſecond.

Or $\underset{\prime}{c}=\dfrac{\grave{m}}{1-ccsc}+\dfrac{\frac{1}{2}\underset{\backprime}{c}^{2}es,\acute{c}}{1-ccsc}$ nearly or $\underset{\backprime}{c}=\dfrac{\grave{m}}{1-ccs,c}$ but $1-ecs,\grave{c}$=the planets diſtance from the ſun. Hence Machin's rule. The error of $\underset{\backprime}{c}$ from neglecting $\dfrac{\frac{1}{2}\underset{\backprime}{c}^{2}cs,\acute{c}}{1-ccs,\grave{c}}$ muſt always be very ſmall, becauſe as was ſhewn above

c is

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* But it ought to be obſerved that a ſmall error, in the excentric anomaly, occaſions a great error in the true anomaly computed from thence, when the orbit is very excentric, and the body near perihelion.

$\underset{\backprime}{c}$ is never greater than $\frac{1,2}{57,2}$ and the errors after each operation will be nearly proportional to $\underset{\backprime}{c}^{2}$, $\underset{\backprime}{c}^{4}$, $\underset{\backprime}{c}^{8}$, &c. But the limit very nearly of the error of $\underset{\backprime}{c}$ after the firſt operation may be eaſily obtained. For the error of $\overset{\backprime}{c}$ is leſs than $\frac{\sqrt{10}}{280} s^{2} \times t^{3},a$ therefore the limit of the error of $\underset{\backprime}{c} = \frac{1}{56 \times 280} \times s^{4} \times t^{6}, a \times \frac{s, c}{1 - cs, c}$ nearly. This quantity is a maximum when $c = 169°$ nearly, in which caſe the quantity itſelf $= ,0000152 = 3''$ nearly. Actual computation ſhews the error in this caſe to be $2''$ nearly.

Hence the following obſervations on Machin's ſolution of Kepler's problem.

1. His method of obtaining a firſt approximation is general for all elliptic orbits whatever, and will give the excentric anomaly always ſufficiently near for correcting it by his ſecond rule. The greateſt error is 1°. 13′½, viz. when the ellipſe is evaneſcent, and the mean anomaly = 180°. When the excentric anomaly is leſs than 23° the uſe of the ſecond rule is unneceſſary, for then the firſt approximation gives the excentric anomaly to leſs than a ſecond.

2. It will be very rarely neceſſary to repeat the ſecond rule, for the excentric anomaly found by one operation will never err more than 2″ from the truth, and that only in the extreme caſe, viz. when the ellipſe is evaneſcent and the excentric anomaly 169°.

3. This ſolution, therefore, of Kepler's problem is complete, but the practice of it, particularly as given by the author, is not ſo convenient in the planetary orbs as other methods.

4. In the extreme caſe, viz. when $c = 1$ and therefore the ellipſe is evaneſcent, this mode of ſolution becomes ſimpler than in any other caſe, for

then ſine $a=\frac{\sqrt[3]{6m}}{\sqrt{10}}$ and the firſt approximated excentric anomaly$=a\sqrt{10}$

This method may be rendered much more convenient for practice in the following manner.

The cubic equation to be ſolved is

$$s^3+\frac{6.\overline{1-e}}{\overline{n^2-1}.e+1}s-\frac{6m}{n.\overline{n^2-1}\;\overline{e+1}}=0$$

This equation is readily ſolved by logarithms (ſee page 57 of Dr. Maſkelynes excellent and uſeful introduction to Taylor's logarithms,) and the following practical rule deduced.

Compute log. $n(=\sqrt{5+\sqrt{\frac{9}{e}+16}})$ which call A

the log. of $\frac{3\sqrt{\overline{n^2-1}\;\overline{e+1}}}{57,296\,n.\overline{1-e}^{\frac{3}{2}}}$ which call B

the log. of $\sqrt{\frac{8.\overline{1-e}}{\overline{n^2-1}\;\overline{e+1}}}$ which call C

Theſe logarithms are conſtant for a given orbit and the computation of them will be facilitated by obſerving that $n=\sqrt{5+4\,sec,\,a}$ a being an arc the cotangent of which is $\frac{1}{4}\sqrt{e}$ and $\sqrt{\overline{n^2-1}\;\overline{e+1}}=$ſec. b, b being an arc the tangent of which is $\sqrt{\overline{n^2-1}\,.\,e}$.

Then m being the degrees in the mean anomaly.

Log. $m+\mathrm{B}+10=$log. t, z. $\frac{\log.\ t,\ \overline{45^\circ-\frac{1}{2}z}+20}{3}=$log. t, u.

log s (ſine a)$=\mathrm{C}+$log. ct. $2u$

log. approximated excentric anomaly $=$log. $a+\mathrm{A}$. Four places of logarithms will be ſufficient and the arcs may be taken out to the neareſt minute.

On

On Dr. Matthew Stewart's Method.

His firſt approximation is as before mentioned, the ſame as Caſſini's. Let AD be the firſt approximated excentric anomaly, and AB the mean Fig. 5. 1. anomaly; then it readily appears that the area DSG = the ſegment BGD AC, being the excentric anomaly. Inſtead of double the former area, Stewart takes its near value DG × perpendicular let fall from S on the tangent at D, and thence deduces a correction to his firſt approximation, viz.

$$\mathrm{DG} : \mathrm{BD}\text{-}s,\ \mathrm{BD} :: t,\ \mathrm{ODS} : s,\ \mathrm{BOD}.$$

To find the error of DG reſulting from this proportion. Draw DL and $\mathrm{G}m$ perpendicular to the tangent DC, and SL and GR parallel to the Fig. 5. 2. ſame. Now the area $\mathrm{DGS}=\mathrm{DSC}-\mathrm{GDC}=\mathrm{LCD}-\mathrm{GDC}$, and $\mathrm{LCD}=\frac{1}{2}\mathrm{LD}\times\mathrm{CD}=\frac{1}{2}\mathrm{LD}\times\overline{\mathrm{GR}-\mathrm{C}m}$, but $\mathrm{C}m=\mathrm{RD}\times t,\ m\mathrm{G}c=\frac{\mathrm{GD}^{2}}{2}\times t,m\,\mathrm{G}c$ (AC being unity) omitting the powers of GD above the ſecond, alſo $\mathrm{GR}=\mathrm{GD}$ omitting the powers of GD above the ſecond. Therefore $\mathrm{LCD}=\frac{1}{2}\mathrm{LD}\times\mathrm{GD}-\frac{1}{2}\mathrm{LD}\times\frac{1}{2}\mathrm{GD}^{2}\times t,\ m\mathrm{G}c$. The area GCD, as is eaſily ſhewn, depends upon the powers of GD above the ſecond; hence, omitting the powers of GD above the ſecond, $\mathrm{DGS}=\frac{1}{2}\mathrm{LD}\times\overline{\mathrm{GD}-\frac{1}{2}\mathrm{GD}^{2}\times t,\ m\mathrm{GC}}$. Therefore inſtead of GD being found by the above proportion, the quantity $\mathrm{GD}-\frac{1}{2}\mathrm{GD}^{2}\times t,\ m\mathrm{GC}$ is deduced. It has been ſhewn before that $\mathrm{GD}=\frac{1}{6}e^{3}s^{3},m+$&c. Hence it follows that the firſt term of the ſeries expreſſing the error of DG will be $\frac{e^{6}}{36}s^{6},m\times t,m\mathrm{GC}$. It is eaſily ſhewn that the tangent of $m\mathrm{GC}$ may be of the ſame order as *e*. Therefore the error of Stewart's corrected approximation depends only in the ſeventh and higher powers of the excentricity.

Hence this method gives the excentric anomaly with great preciſion for all the planets, and even for orbits more excentric; but the practice of the method is leſs ſimple than Caſſini's firſt approximation corrected by Newton's firſt method, or by the method hereafter given. It alſo does not readily afford a farther approximation.

* *On Simpſon's practical Rule.*

Simpſon having deduced a ſeries for the equation, finds the ſum of that ſeries, as far as the third power of the excentricity incluſive, nearly equivalent to an arch, the ſine of which is $\frac{2es,\ m}{1+\frac{5}{4}ecs,\ m}$ diminiſhed by $\frac{1}{3}$ of the arch the ſine of which is $\left(\frac{2es,\ m}{1+\frac{5}{4}ecs,\ m}\right)^3$. The practical rule thence ariſing is full as eaſy as the rule by the firſt approximation of Caſſini; but then no convenient method offers itſelf for extending the approximation by Simpſon's method. Simpſon's anomaly computed by this rule, eaſily appears to be, regarding only the third power of the excentricity.

$$m - 2es,\ m + \frac{5}{4}e^2 s,\ 2m - \frac{7}{32}e^3 s,\ m + \frac{107}{96}e^3 s,\ 3m$$

comparing this with the ſeries for the true anomaly, the correction appears to be $-\frac{1}{32}e^3 s,\ m - \frac{1}{32}e^3 s,\ 3m$, which is a maximum when $m = 35° \ 16'$ nearly. And therefore in the orbit of Mercury the error from this term when a max. is about 57″, in the orbits of the other planets the errors are as the author rightly obſerves very ſmall.

On

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* Eſſays, p. 47.

*On Lacaille's Method.**

The indirect method uſed by Lacaille, and recommended by Lalande, is as follows.

A ſuppoſition is made for the true anomaly, with this ſuppoſed true anomaly the excentric anomaly is computed by the analogy.

$$\sqrt{1-e} : \sqrt{1+e} :: t, \tfrac{1}{2}\text{ anom.} : t, \tfrac{1}{2}\text{ excent. anom.}$$

From the excentric anomaly ſo deduced, the mean anomaly is computed by the equation $m = c + es, c$. The error of this mean anomaly is applied as a correction to the aſſumed anomaly, and the operation repeated till no error remains in the computed mean anomaly. The proper mode of eſtimating the value of this method ſeems to be to enquire how the repeated operations converge.

To aſcertain the rate of convergency, we may make uſe of the equation $a = m - 2cs, m +$ &c. Let $\acute{m}$ be the mean anomaly, computed to the aſſumed anomaly $\acute{a}$, ſuppoſing a nearly equal $\acute{a}$.

Then

$$a - \grave{a} = m - \grave{m} - \overline{2cs, m - s, \grave{m}}\ \text{\&c.} = m - \grave{m} \times \overline{1 - 2ecs\grave{m}},\ \text{\&c.}$$

ſo that the error of $\grave{a}$ differs from the error of $\acute{m}$

only by $\overline{m - \grave{m}} \times 2ecs, m$

Hence the convergency proceeds according to the ſimple power of the excentricity, and if the mean anomaly be the firſt aſſumption, the error of the computed mean anomaly is of the order of the firſt power of the excentricity. Therefore five or ſix operations may be neceſſary for Mercury or Pallas, and in orbits ſtill more excentric a greater number. Nor ſhould we gain much in point of brevity, by previouſly computing by Ward's or Boulliald's hypotheſis, or even by Simpſon's practical rule, the true anomaly, nearly for the firſt aſſumption. It muſt therefore be concluded that although this rule is very convenient and ſimple in practice, yet it yields to Caſſini's method, even when corrected by Kepler's method. Both theſe methods in orbits of great excentricity converge ſlowly. The following practical method is free from that inconvenience.

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* Mem. Acad. 1750. Lalande's Aſtr. Lalande edit. Halley's Tab. vol. 2, 50.

A convenient practical Method of Solution.

It has been ſhewn that the error of Caſſini's firſt approximation depends only on the third and higher powers of the excentricity. So that in orbits of ſmall excentricity, the firſt approximated is very near the true excentric anomaly. If $\underset{\backslash}{c}$ be the error of this approximation and $\overset{\backslash}{c}$ the approximated excentric anomaly itſelf, alſo $\overset{\backslash}{m}$ the mean anomaly computed with the excentric anomaly $\overset{\backslash}{c}$; then as has been ſhewn

$\underset{\backslash}{c}=\dfrac{\overset{\backslash}{m}}{\overset{\backslash}{1}+ccs,\ c}$ nearly and the error of c thus computed is nearly

$\dfrac{\frac{1}{2}c^2es,\overset{\prime}{c};}{1+ecs,\overset{\backslash}{c}}$ therefore $\underset{\backslash}{c}$ depending only on the third power of the excentricity the error of $\underset{/}{c}$ will depend only on the ſeventh &c. Hence in orbits of ſmall excentricity the excentric anomaly is readily obtained. And even in orbits of the greateſt excentricity, Caſſini's firſt approximation is ſufficient to apply the above formula for a further correction. For taking the extreme caſe, viz. when $c=1$; $\overset{\backslash}{m}+\underset{\backslash}{m}=\overset{\backslash}{c}+\underset{/}{c}+s,\ \overline{\overset{\backslash}{c}+\underset{/}{c}}$ $\overset{/}{m}=\overset{\backslash}{c}+s,\ \overset{\backslash}{c}$. From whence is readily deduced

$$\underset{/}{m}=\underset{/}{c}+cs,\overset{\backslash}{c}\times\overline{\underset{/}{c}-\tfrac{1}{6}\underset{/}{c}}\ \overline{+\&c.}-s,\ \overset{\backslash}{c}\times\overline{\tfrac{1}{2}\underset{\backslash}{c}^2-\tfrac{1}{24}\underset{/}{c}^4}\ \overline{+\&c.}$$

Now the firſt approximated excentric anomaly with this excentricity is half the mean anomaly, and its greateſt error will be when the mean anomaly $=180°$. In that caſe $\underset{/}{c}=\dfrac{3,14159}{2}=1,57079$, with which value of $\underset{\backslash}{c}$ both the ſeries $1-\tfrac{1}{6}\underset{\backslash}{c}^2+\&c.$ $\tfrac{1}{2}\underset{\backslash}{c}-\tfrac{1}{24}\underset{\backslash}{c}^3+\&c.$ will converge ſo that a near value of $\underset{/}{c}$ may be derived from the equation $\underset{/}{m}=\underset{/}{c}+cs,\overset{/}{\underset{\backslash}{c}}$ or from $\underset{/}{c}=\dfrac{\underset{/}{m}}{1+cs,c}$; But when $m=180°$ the approximated excentric anomaly will continually approach to 180° alſo, and conſequently the denominator $1+cs\overset{\backslash}{c}$ become evaneſcent. In this extreme caſe the formula from the ſimple equation fails, but a formula might if it were worth while be here obtained from the quadratic equation $\grave{m}=\underset{\backslash}{c}\times 1+cs\grave{c}-\frac{1}{2}\,\underset{/}{c}^{2}\,s,\grave{c}$. In this manner, therefore, a complete ſolution of Kepler's problem may be obtained, and very conveniently, except when the ellipſe is evaneſcent, and at the ſame time the mean anomaly nearly 180°*.

When the ellipſe is very excentric or nearly evaneſcent, and the mean anomaly nearly 180°, a ſmall error in the excentric anomaly occaſions a great error in the true anomaly. Hence an inconveniency in this caſe, in deriving from the excentric anomaly, computed by *any method*, the true anomaly. This takes place with reſpect to comets, when they are near perihelion or are viſible to us. And therefore, for them, when near perihelion, inſtead of uſing the following practical rule, the beſt method is to derive by a correction the true anomaly, reckoned from perihelion from the anomaly in a parabola, having the ſame perihelion diſtance.

The Log. of the multiplier $\frac{1}{1+ecs,\grave{c}} = 20 - \log.\ 2 - 2\ \log.\ cs\tfrac{1}{2}\alpha$,

α being an arch the log. coſine of which is the Log. e + Log. $cs,\ \acute{c}$. It appears to me rather more convenient in practice to compute previouſly a ſmall table containing the logarithms of the above multipliers, which may be done very expeditiouſly for a given orbit. In the moſt excentric planetary orbits, if the table be computed to every five degrees of excentric anomaly, and the logarithms to five places of figures, it will be ſufficient. That a compariſon may be made of the practice of the method with, and without the tables, the table is ſubjoined for the new planet Pallas diſcovered by Dr. Olbers.

What has been ſaid will, I think, ſufficiently explain the convenience and extent of the following practical rule.

Practical

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* This reaſoning may be eaſily applied to Newton's firſt ſolution, and in the examination of which it might have been inſerted, but it was thought better to place it here, becauſe Caſſini's ſolution was poſterior to Newton's.

PRACTICAL RULE

FOR

OBTAINING THE TRUE FROM THE MEAN ANOMALY.

1. *Subtract the Log. of the aphelion from the Log. of the perihelion diſtance, and call the remainder* B. B + *the Log. tang. of* $\frac{1}{2}$ *the mean anomaly* = *the Log. tang. of an arch, which being added to* $\frac{1}{2}$ *the mean anomaly, the ſum will be the approximate excentric anomaly.*

2. *Add together the conſtant Log.* 5.3144251, *the Log. of the excentricity and the Log. ſine of the computed approximate excentric anomaly: the ſum diminiſhed by* 10 *will be the Log. of a number of ſeconds to be added to the approximate excentric anomaly, to obtain the correſponding mean anomaly.*

3. *Find the logarithm C either by a table previouſly conſtructed, or by computation, viz.* $C = 19.6989700 - 2$ *Log. cs* $\frac{1}{2}\alpha$, α *being an arch the Log. coſine of which* = *Log. excent* + *Log. cs of the approximated excentric anomaly. When the approximated excentric anomaly is greater than* 90° α *is alſo greater than* 90°.

3. *Add the log. of the error of the computed mean anomaly to C, and the ſum will be the logarithm of the error of the firſt approximated excentric anomaly. This error has always the ſame ſign as the error of the computed mean anomaly.*

5. *The ſum of* $\frac{1}{2}$ B *and Log. tang.* $\frac{1}{2}$ *the correct excentric anomaly is the Log. tang. of* $\frac{1}{2}$ *the true anomaly.*

Note, when the mean anomaly is greater than 180°. *Take its ſupplement to* 360, *and compute the correſponding true anomaly: its ſupplement to* 360 *will be the true anomaly required.*

By the above operations, the true anomaly will be had to leſs than a ſecond, for all the planets. In orbits more excentric, it may be neceſſary to repeat the ſecond, third and fourth rules, uſing the corrected excentric anomaly inſtead of that firſt computed. And then uſing the ſecond corrected excentric anomaly to obtain a third, and ſo on till no error remains. The ſucceſſive corrected excentric anomalies will rapidly approximate to the true excentric anomaly, as has been before, ſhewn.

Although a ſecond operation will never be neceſſary for the planets, yet as a ſecond operation is very readily performed, it may be uſed for two reaſons. 1. It ſerves as a check, to verify the corrected excentric anomaly, and 2. when the ſine of the corrected excentric anomaly is taken out from the logarithmic tables, the coſine of the ſame may be taken out, which will readily give us the logarithm of the planet's diſtance from the ſun, viz. 2 Log. cs, $\frac{1}{2}\alpha - 19.698700$ α being the arch, the Log. cs of which $=$ Log. excentricity $+$ Log. cs corrected excentric anomaly.

TABLE FOR THE PLANET PALLAS,

Approx. Exc. An.	Log. = C	Approx. Exc. An.	Log. = C	Approx. Exc. An.	Log. = C
0°	$\bar{1}$. 89997	65°	$\bar{1}$. 95489	130	0. 07909
5	$\bar{1}$. 90031	70	$\bar{1}$. 96315	135	0. 08785
10	$\bar{1}$. 90133	75	$\bar{1}$. 97183	140	0. 09605
15	$\bar{1}$. 90303	80	$\bar{1}$. 98089	145	0. 10357
20	$\bar{1}$. 90539	85	$\bar{1}$. 99031	150	0. 11031
25	$\bar{1}$. 90843	90	0. 00000	155	0. 11519
30	$\bar{1}$. 91211	95	0. 00991	160	0. 12113
35	$\bar{1}$. 91643	100	0. 01999	165	0. 12503
40	$\bar{1}$. 92139	105	0. 03013	170	0. 12789
45	$\bar{1}$. 92697	110	0. 04029	175	0. 12961
50	$\bar{1}$. 93313	115	0. 05035	180	0. 13009
55	$\bar{1}$. 93985	120	0. 06023		
60	$\bar{1}$. 94711	125	0. 06985		

EXAMPLE I.

To find the true Anomaly of Pallas, when the mean Anomaly = 45°, and also the Logarithm of the Planet's distance from the Sun.

The excentricity of the orbit of Pallas = 0, 259, the mean distance being unity.

Log.

Log. per. dist. = ,741	$\bar{1}.8698182$	31552'',4 =	8°. 45' 52'', 4
Log. aph. dist. = 1,259	0. 1000257		36. 12. 3
Log = B	1. 7697925	Compᵈ. m. anom.	44. 57. 55, 4
			45
Log. tang. ½ 45° = 22°. 30' -	9. 6172243	Error -	2. 4, 6
Log. tang. 13°. 42' 3'', 3	9. 3870168		124'', 6
Add 22. 30			
Log. sine 36 12 3, 3	9. 7713071	Log. *cs* 36°. 12'. 3''	9. 90685
Log, 259	$\bar{1}.4132998$	Log, 259	$\bar{1}.41330$
'' Constant Log.	5. 8144251	Log. *cs* 77 56. 10	9. 32015
Log. 31552, 4	4. 4990320	Log. *cs* 38. 58. 5	9. 89070
			2
			19. 78140
Log. 124'', 6	2. 09552	Const. Log.	19. 69897
C	$\bar{1}.91762$	Log. = C	$\bar{1}.91757$
Log. 103'', 1	2. 01314	Or by the Table C = $\bar{1}.91762$	

36° 12' 3'', 3
+ 1. 43, 1

36. 13 46, 4 Corrected excent. anom.

½ B	$\bar{1}.8848962$
t. 18°. 6'. 53'', 2	9. 5147282
t. 14. 5. 19	9. 3996244
2	

t. 28 10 38 true anomaly, of which the error is not half a second.

Or using a second operation, to serve as a check upon the computation; and also, to investigate the accurate logarithm of the planet's distance from the sun.

Log. sine $36^\circ.13'.46'',4$	9.7716035	cs 9.9066881
Log. 259	$\bar{1}.4132998$	$\bar{1}.4132998$
Const. log.	5.3144251	9.3199879
Log. $31573'',9$ or	4.4993284	cs $77^\circ.56'.26'',5$
$8^\circ.46.13,9$		$38\ 5813,2$
$36.13.46,4$ mean anom.		
$45\quad 0.\ 0,3$	differing $\frac{3}{10}''$ from the truth.	

cs $38^\circ.58'.13'',2$	9.8906846
	2
	$19\ 7813692$
	19.6989700
Log. planet's dist.	0.0823992

Example 2. The ellipse being evanescent, and the mean anomaly$=150^\circ$, required the excentric anomaly.

Here the perihelion distance being evanescent, by the rule the first approximated excentric anomaly$=\frac{1}{2}$ the mean anomaly, or 75°.

Log. sine 75°	9.98494	
Const. Log.	5.31442	
Log. $199230''$	5.29936	
$=55^\circ.20'$		
75		
$130.\ 20$	comp. mean anom.	
150		
$19.40.=1180'$	Error	

The computation of C

Log. cs 75°	9.89947
2	2
	19.79894
Const. log.	19.69897
C.	$\bar{1}.90003$
Log. $1180'$	3.07138
Log. 937	2.97191
$=15^\circ.37'$	corⁿ. ex. anom.
75	
$2 \mid 90.37$	1st. corᵈ. ex. anom.
45.18	

2d. Operation.

2d. Operation.

Log. sine 90°. 37′	9. 99997
Const. log.	5. 31442
Log. 206250″	5. 31439
= 57°. 17′	
90. 37	
147. 54	
150	
2. 6 = 126 Error	

Log. *cs* 45°. 18′	9. 84720
	2
	19. 69440
Const. log.	19. 69897
C	0. 00457
Log. 126″	2. 10037
Log. 127	2. 10494
= 2. 7	
90. 37	
2 | 92. 44 2d. corr. ex. anom.	
46. 22	

3d. Operation.

Log. sine 92. 44	9. 9995056
Const. log.	5. 3144251
Log. 206030	5. 3139307
= 57. 13. 50	
92. 44	
149. 57. 50	
150	
2. 10 = 130″ Error	

Log. *cs* 46. 22	9. 83887
	2
	19. 67774
	19. 69897
C	0. 02123
Log. 130″	2. 11394
Log. 136	2. 13517
= 2′. 16″	
9. 44	
92. 46. 16 3d cor^{d}. ex. anom.	

This last corrected excentric anomaly is true to a second.

The preceding example is not given as one that can occur in practice, or as the ſhorteſt method, in this caſe,* of finding the excentric anomaly, but merely to ſhew, that in the extreme caſe of excentricity the method holds. It only fails, as was obſerved, when the ellipſe is nearly evaneſcent, and at the ſame time the mean anomaly nearly 180°. Machin's method gives, in that instance, the excentric anomaly with very great preciſion. But, as was before remarked, in very excentric orbits, when the body is near perihelion, it is very inconvenient to inveſtigate the true anomaly by means of the excentric anomaly, becauſe a ſmall error in the excentric anomaly occaſions a great error in the true anomaly. And, although Machin's method furniſhes us with the excentric anomaly, in that caſe, with as great a degree of accuracy as can be deſired, yet the common tables of logarithms are not extended to a ſufficient number of places to uſe them for that purpoſe.

The beſt method of ſolving this caſe, which is that of the planets when viſible to us, is to deduce the true anomaly meaſured from perihelion, from the correſponding anomaly in a parabola, having the ſame perihelion diſtance.

I had not intended to notice this method farther; but as the following ſolution appears to be as ſimple as can be deſired, alſo readily admits of being extended to any degree of accuracy, and at the ſame time affords another ſtriking illuſtration of the convenience of the method of deriving ſeries above alluded to, I have been induced to add it. T. Simpſon and Laplace are the only authors I have met with who have ſolved this problem.† Their ſolutions give only the firſt term of the correction to be applied to the anomaly in the parabola. In the following, two terms of the correction are given, and more might be eaſily deduced, if neceſſary.

Prob. Having given the anomaly in a parabola reckoned from perihelion, to deduce from thence the correſponding anomaly in an excentric ellipſe, having the ſame perihelion diſtance. The anomalies are called correſponding, when the times from perihelion in each orbit are the ſame.

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* In and near the caſe of extreme excentricity, the common tables of natural ſines and circular arcs will readily furniſh, by mere inſpection, the excentric anomaly true to the neareſt degree; and then two operations will find it true to a ſecond.

† P. 58, 60. Miſc. Tracts.——Mécanique céleſte, Tom. 1. p. 183, 186.

Solution. Let $a = \frac{1}{2}$ the true anomaly in the ellipſe to radius unity $A = \frac{1}{2}$ the correſponding anomaly in the parabola, the perihelion diſtance $= 1$ and the ſemi axis major of the ellipſe $= \alpha$ Then the fluxion of the elliptic area from perihelion $= \dot{a} \times \overline{\text{diſt.}}\big|^{2} = \dot{a} \times \overline{\frac{2\alpha - 1}{\alpha + \overline{\alpha - 1}\,cs,2a}}\Big|$

$$= \dot{a} \times \overline{\frac{2-n}{1+cs,2a}}\Big|^{2} \times \frac{1}{\overline{1 - \frac{n\,cs,2a}{1+cs,2a}}\Big|^{2}} \left(\text{putting } n = \frac{1}{\alpha}\right) = \dot{a} \times \overline{\frac{2-n}{2cs,^{2}a}}\Big|$$

$$\times \frac{1}{\overline{1-n+\frac{n}{2cs,^{2}a}}\Big|^{2}} = \dot{a} \times \overline{\frac{2-n}{2-2n}}\Big|^{2} \times : \frac{1}{cs,^{4}a} - \frac{2}{2} \times \frac{n}{1-n} \times \frac{1}{cs,^{6}a} + \frac{3}{4} \times$$

$\overline{\frac{n}{1-n}}\Big|^{2} \times \frac{1}{cs,^{8}a} -$ &c. Alſo, the fluxion of the parabolic area $= \frac{\dot{A}}{cs,^{4}A}$ Now the ratio of the area in the ellipſe, to the cotemporaneous area in the parabola, is the ſubduplicate ratio of the parameters, or of $\sqrt{2-n} : \sqrt{2}$ *

$$\text{Hence, } \overline{1-n}\big|^{2} \times \overline{1-\tfrac{1}{2}n}\Big|^{-\frac{3}{2}} \times fl.\,\frac{\dot{A}}{cs,^{4}A} = fl.\,\frac{\dot{a}}{cs,^{4}a} - \frac{n}{1-n} \times \frac{2}{2} fl.\,\frac{\dot{a}}{cs,^{6}a} +$$

$$\overline{\frac{n}{1-n}}\Big|^{2} \times \frac{3}{4} fl.\,\frac{\dot{a}}{cs,^{8}a} - \text{\&c.}$$

Let the ſucceſſive fluxions of this equation be taken *per Saltum*, making n flow uniformly, $a = A$ and $n = o$:

1.

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* Newtoni Prin. Prop. 14. Lib. 1.

$$1.\qquad -\frac{5}{4}\dot{n}\times\text{fl.}\,\frac{\dot{A}}{cs,^4A}=\frac{\dot{a}}{cs,^4A}-\dot{n}\,\text{fl.}\,\frac{\dot{A}}{cs,^6A}$$

$$2.\qquad -\frac{1}{16}\dot{n}^2\times\text{fl.}\,\frac{\dot{A}}{cs,^4A}=\frac{\ddot{a}}{cs,^4A}+\frac{4\dot{a}^2s,A}{cs,^5A}-\frac{2\dot{n}\dot{a}}{cs,^6A}-2n^2\,\text{fl.}\,\frac{\dot{A}}{cs,^6A}$$

$$+\frac{3}{2}\dot{n}^2\,\text{fl.}\,\frac{\dot{A}}{cs,^8A}\ \text{\&c. \&c.}$$

The $\text{fl.}\,\frac{\dot{A}}{cs,^4A}=\frac{s,A}{3cs,^3A}+\frac{2s,A}{3cs,A}$

$$\text{fl.}\,\frac{\dot{A}}{cs^6A}=\frac{s,A}{5cs,^5A}+\frac{4s,A}{5.3cs^3A}+\frac{4.2sA}{5.3cs,A}$$

$$\text{fl.}\,\frac{\dot{A}}{cs,^8A}=\frac{s,A}{7cs,^7A}+\frac{6s,A}{7.5cs,^5A}+\frac{6.4s,A}{7.5.3cs^3A}+\frac{6.4.2sA}{7.5.3cs,A}$$

Whence, from the above equations we readily deduce $2a$ (the anomaly in the ellipſe) $=2A+2\times\overline{\dot{a}+\frac{\ddot{a}}{2}}=2A+\frac{t,A}{10}\times\overline{4-3cs,^2A-6cs,^2A}\times n+\frac{t,A}{2800cs,^2A}$ $\times\overline{408-160cs,^2A-1100cs,^4A-425cs^6A+252cs^{10}A}\times n^2$. It may be obſerved, that if the axis major of the ellipſe be unity, and the excentricity $=e$, $n=1-e$ the perihelion diſtance. The co-efficient of n is preciſely the ſame as that found by Laplace,* by a very different method. Q. E. I.

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* Mécanique Céleſte Tom. 1. p: 186

What has been done in this eſſay has been principally done with a view of comparing different ſolutions of Kepler's Problem. That compariſon has led me to point out what I conſider as the beſt practical ſolution of the problem, particularly applicable to the planets. This ſolution is formed by a combination of the ſolutions of Kepler, Newton, and Caſſini. The very ſmall ſhare I claim in it is from having recommended that combination of ſolutions. The ſolutions of the two latter have been ſeparately recommended by writers on aſtronomy. Caſſini has not always been referred to as the author of his method, and Newton rarely. The merit of Caſſini's method is derived from its ſimplicity, and ready application to the planetary orbits. Newton's ſolution was the firſt that was applicable to orbits of every degree of excentricity.

In addition to the ſolutions that have been mentioned, it is neceſſary conſiſtently with my plan, to notice two others. The one given by Herman, in 1725,* and the other by Mr. Ivory.†

The ſolution of M. Herman had, through inadvertency, eſcaped my notice, although referred to by ſeveral authors. And is given in the ſame memoir, together with a conſtruction of the problem. It is in ſubſtance the ſame as the ſolution of Dr. Matthew Stewart, that has been examined, and therefore what has been ſaid of that may ſuffice for Herman's ſolution. The ſolutions only differ by Stewart taking D × perpend. from S on the tang. at D = twice the area DSG, and by Herman's taking SD × perpend. from
Fig. 5. 1 G on SD for the ſame area. Herman uſes Caſſinis approximation without reference, although he had mentioned that ſolution in the beginning of his memoir. Dalembert, alſo, in the Encyclopédie, juſtly commending Herman's Solution, does not notice that the moſt valuable part of it was due to Caſſini.

I had not an opportunity of ſeeing the ſolution of Mr. Ivory till this eſſay was nearly printed. It is not, however, neceſſary for me to enter into a minute examination of it; as the ingenious author has very fully explained

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Comment. Acad. Petrop. Tom. 1, p. 142.

Edinburgh Tranſactions, Vol. 5. part ii.

plained the principles and extent of his method. With regard to its practical value for the planetary orbits, it yields to ſeveral of the methods that have been mentioned, and with reſpect to elliptical orbits in general, it certainly yields to Newton's method, and perhaps to Machin's, Mr. Ivory remarks the ſame inconvenience in his method as was obſerved above with reſpect to Newton's; the computation of the excentric anomaly in orbits very excentric, when the body is near perihelion. This inconvenience does not exiſt in Machin's ſolution; in that part of the orbit his firſt approximation is as exact as can be deſired. In the extreme caſe when the ellipſe is evaneſcent, the ſolution derived from Newton's method is much more ſimple than that of Mr. Ivory. And alſo, in that caſe, Machin's ſolution is more commodious than that in the Edinburgh Tranſactions. It is with concern I have made theſe remarks on the labours of a perſon who has* merited ſo much by his moſt elegant and uſeful ſolution of a problem connected with phyſical aſtronomy. A problem on which the eminent mathematicions of Europe had neceſſarily exerted their ingenuity for nearly half a century; and whoſe ſolutions have all been ſurpaſſed by that of Mr. Ivory. In his ſolution of Kepler's problem, he has added the method of deriving the place of a comet, *moving in an excentric ellipſe from* the place in a parabola having the ſame perihelion diſtance. He conſiders the problem as new, although, beſide Simpſon and Laplace above referred to, Lalande mentions the problem. Mr. Ivory has given two terms of the ſeries derived by his method.

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* Edinb. Tranſactions, Vol. 4. 1798.